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STRONGLY RESOLVABLE (r, λ)-DESIGNS

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The aim of this paper is to show the following Theorem: If D is an (r, λ)-design (regular pairwise balanced design) with a resolution having c classes, then $b+1 \geq v+c$. Equality holds if and only if the number of points on two distinct blocks depends only on the classes, the two blocks belong to. Resolvable (r, λ)-designs with $b+1=v+c$ are called strongly resolvable. Using symmetric block designs we shall construct many strongly resolvable (r, λ)-designs.

1. Introduction

An (r, λ)-*design* is an incidence structure $\mathsf{D}=(\mathscr{P}, \mathscr{B}, I)$ satisfying the following axioms:

– Any point of D is incident with exactly r blocks.

– Any two distinct points are connected by exactly λ blocks of D.

– There exist at least two blocks, any block has at least two points, outside any block there are at least two points.

Let $\mathsf{D}=(\mathscr{P}, \mathscr{B}, I)$ be an (r, λ)-design with v points and b blocks. A *resolution* of D is a partition $\{\mathscr{B}_1, \ldots, \mathscr{B}_c\}$ of $\mathscr{B}$ in *classes* $\mathscr{B}_i$ such that any point of D is incident with a constant number $\rho_i>0$ of blocks of the class $\mathscr{B}_i$ $(i \in \{1, \ldots, c\})$. The positive integers $\rho_1, \ldots, \rho_c$ are called the *parameters* of the resolution $\{\mathscr{B}_1, \ldots, \mathscr{B}_c\}$. If D is resolvable with $\rho_1=\cdots=\rho_c=\rho$, then D is said to be ρ-*resolvable.*

The most interesting (r, λ)-designs are the $2-(v, k, \lambda)$ *block designs*, which can be defined as those (r, λ)-designs in which any block is incident with the same number k of points. Resolutions of block designs have been thoroughly investigated. In this paper, we shall mainly deal with the following generalization of a theorem due to Hughes and Piper [2]:

Theorem 1. *Denote by* D *an* (r, λ)*-design with a resolution* $\{\mathscr{B}_1, \ldots, \mathscr{B}_c\}$. *If* v *is the number of points and* b *the number of blocks of* D, *then* $b+1 \geq v+c$. *Equality holds if and only if there exist integers* μ_{ij} *such that any two distinct blocks* B *and* C *with* $B \in \mathscr{B}_i$ *and* $C \in \mathscr{B}_j$ *intersect in exactly* μ_{ij} *points* $(i, j \in \{1, \ldots, c\})$.

Resolvable (r, λ)-designs with $b+1=v+c$ are called *strongly resolvable.* In Section 2 we prove $b+1 \geq v+c$ by a method which was used by Vanstone [4] for the investigation of 1-resolvable (r, λ)-designs. In Section 3 we shall study strongly resolvable (r, λ)-designs by a similar method as in Hughes and Piper [2]. Finally, in Section 4, we show the following

Theorem 2. *Any strongly ρ-resolvable (r, λ)-design in which no block is incident with more than $\frac{1}{2}v$ points is a strongly resolvable block design.*

This generalizes Theorem 4.2 of Vanstone [4] who proved it for $\rho = 1$. We shall see, however, that this theorem does not generalize to arbitrary strongly resolvable (r, λ)-designs: Using symmetric block designs we construct a variety of strongly resolvable (r, λ)-designs with two block lengths, both of which are smaller than $\frac{1}{2}v$.

2. Resolvable (r, λ)-designs

Throughout this paper we shall use the terminology of Dembowski [1]. In particular, for two blocks B and C the number of points which are incident with B and C is denoted by $[B, C]$.

Proposition 1. *If $\mathsf{D} = (\mathfrak{p}, \mathcal{B}, I)$ is an (r, λ)-design with a resolution $\{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$, then $b + 1 \geq v + c$.*

Proof. Let $\rho_1, \ldots, \rho_c$ be the parameters of the resolution $\{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$. We number the points of D in some order: $\mathfrak{p} = \{p_1, \ldots, p_v\}$. Moreover, let $B_{i1}, \ldots, B_{im_i}$ be the blocks in the class $\mathcal{B}_i$ $(i \in \{1, \ldots, c\})$.

We consider the free vector space V over $\mathfrak{p} \cup \{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$. Define the vectors $\bar{B}_{ij}$ and B^+ as follows:

$$\bar{B}_{ij} = \sum_{p_s I B_{ij}} p_s + \mathcal{B}_i \qquad (i \in \{1, \ldots, c\}, j \in \{1, \ldots, m_i\})$$

and

$$B^+ = \sum_{i=1}^{c} \rho_i \mathcal{B}_i.$$

Clearly, it suffices to show that the set

$$\mathcal{B}^+ = \{B^+\} \cup \{\bar{B}_{ij} \mid i \in \{1, \ldots, c\}, j \in \{1, \ldots, m_i\}\}$$

spans V. (*Then* $b + 1 = |\mathcal{B}^+| \geq |\mathfrak{p} \cup \{\mathcal{B}_1, \ldots, \mathcal{B}_c\}| = v + c$.)

For $i \in \{1, \ldots, c\}$ it holds

$$\sum_{j=1}^{m_i} \bar{B}_{ij} = \sum_{j=1}^{m_i} \Bigl(\sum_{p_s I B_{ij}} p_s + \mathcal{B}_i \Bigr)$$

$$= \sum_{j=1}^{m_i} \sum_{p_s I B_{ij}} p_s + m_i \mathcal{B}_i = \rho_i \sum_{s=1}^{v} p_s + m_i \mathcal{B}_i. \tag{1}$$

For an arbitrary point p_t of $\mathbf{D}$, let us denote by $B_{i1}, \ldots, B_{i\rho_i}$ the blocks of $\mathcal{B}_i$ through p_t $(i \in \{1, \ldots, c\})$. It follows

$$\begin{aligned}\sum_{p_t I B_{ij}} \bar{B}_{ij} &= \sum_{i=1}^{c} \sum_{u=1}^{\rho_i} \bar{B}_{iu} = \sum_{i=1}^{c} \sum_{u=1}^{\rho_i} \Big(\sum_{p_s I B_{iu}} p_s + \mathcal{B}_i \Big) \\ &= \sum_{i=1}^{c} \sum_{u=1}^{\rho_i} \sum_{p_s I B_{iu}} p_s + \sum_{i=1}^{c} \sum_{u=1}^{\rho_i} \mathcal{B}_i = r p_t + \lambda \sum_{\substack{s=1 \\ s \neq t}}^{v} p_s + \sum_{i=1}^{c} \rho_i \mathcal{B}_i \qquad (2) \\ &= (r-\lambda) p_t + \lambda \sum_{s=1}^{v} p_s + B^+.\end{aligned}$$

If we define the vector M by $M = \sum_{s=1}^{v} p_s$, it follows from (1):

$$\mathcal{B}_i = m_i^{-1} \sum_{j=1}^{m_i} \bar{B}_{ij} - \frac{\rho_i}{m_i} M.$$

Consequently,

$$B^+ = \sum_{i=1}^{c} \rho_i \mathcal{B}_i = \sum_{i=1}^{c} \frac{\rho_i}{m_i} \sum_{j=1}^{m_i} \bar{B}_{ij} - M \sum_{i=1}^{c} \frac{\rho_i^2}{m_i}.$$

Together with

$$L = \sum_{i=1}^{c} \frac{\rho_i^2}{m_i}$$

we get

$$M = \frac{1}{L} \sum_{i=1}^{c} \frac{\rho_i}{m_i} \sum_{j=1}^{m_i} \bar{B}_{ij} - \frac{1}{L} B^+. \qquad (3)$$

Using (2) it follows with (3) that

$$(r-\lambda) p_t = \sum_{p_t I B_{ij}} \bar{B}_{ij} - \lambda M - B^+$$

is a linear combination of elements in $\mathcal{B}^+$. Since $r - \lambda \neq 0$, the vector p_t $(t \in \{1, \ldots, v\})$ is contained in the span of $\mathcal{B}^+$.

Finally, (1) reads

$$m_i \mathcal{B}_i = \sum_{j=1}^{m_i} \bar{B}_{ij} - \rho_i M.$$

So, according to (3), $\mathcal{B}_i$ is in the span of $\mathcal{B}^+$ as well.

Hence $\mathcal{B}^+$ generates V. $\square$

The resolution $\{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$ of the (r, λ)-design $\mathbf{D}$ is called *strong*, if $b + 1 = v + c$ holds; $\mathbf{D}$ is said to be *strongly resolvable*, if $\mathbf{D}$ admits a strong resolution.

Corollary 1. *Denote by* $\mathbf{D}$ *an* (r, λ)*-design with a strong resolution* $\{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$. *Then there exist integers* $k_1, \ldots, k_c$ *such that any block of* $\mathcal{B}_i$ *is incident with exactly* k_i *points* $(i \in \{1, \ldots, c\})$.

Proof. We use the terminology of the proof of Proposition 1 again. Define

$$k_{ij} = [B_{ij}] \qquad (i \in \{1, \dots, c\}, j \in \{1, \dots, m_i\}).$$

Using the equations (2) and (3) we get

$$\begin{aligned}\sum_{i=1}^{c} \sum_{j=1}^{m_i} k_{ij}\bar{B}_{ij} &= \sum_{i=1}^{c} \sum_{j=1}^{m_i} \sum_{p_s I B_{ij}} \bar{B}_{ij} = \sum_{B_{ij} \in \mathscr{B}} \sum_{p_s I B_{ij}} \bar{B}_{ij} \\ &= \sum_{p_s \in \mathscr{P}} \sum_{p_s I B_{ij}} \bar{B}_{ij} = \sum_{s=1}^{v} \sum_{p_s I B_{ij}} \bar{B}_{ij} = \sum_{s=1}^{v} [(r-\lambda)p_s + \lambda M - B^+] \\ &= (r-\lambda) \sum_{s=1}^{v} p_s + v\lambda M + B^+ = (r - \lambda + v\lambda)M + vB^+ \\ &= \frac{r-\lambda+v\lambda}{L}\left(\sum_{i=1}^{c} \frac{\rho_i}{m_i} \sum_{j=1}^{m_i} \bar{B}_{ij} - B^+\right) + vB^+. \end{aligned} \tag{4}$$

Therefore

$$\sum_{i=1}^{c} \sum_{j=1}^{m_i} \left(k_{ij} - \frac{(r-\lambda+v\lambda)\rho_i}{Lm_i}\right)\bar{B}_{ij} + \left(\frac{r-\lambda+v\lambda}{L} - v\right)B^+ = 0.$$

Now, $b + 1 = v + c$ implies that $\mathscr{B}'$ is a basis of V; consequently,

$$k_{ij} = \frac{r-\lambda+v\lambda}{L}\frac{\rho_i}{m_i}.$$

It follows that k_{ij} is independent of j. If we put $k_i = k_{ij}$ for a $j \in \{1, \dots, m_i\}$ ($i \in \{1, \dots, c\}$), Corollary 1 is proved. □

The numbers ρ_i, m_i, k_i are called the *parameters* of the strongly resolvable (r, λ)-design $\mathbf{D}$.

Corollary 2. *For $i \in \{1, \dots, c\}$ we have $\rho_i v = m_i k_i$.* □

3. Strongly resolvable (r, λ)-designs

Proposition 2. *Denote by $\mathbf{D} = (\mathscr{P}, \mathscr{B}, I)$ a strongly resolvable (r, λ)-design with parameters ρ_i, m_i, k_i. Then $\mathbf{D}$ satisfies the following condition:*

(∗) *There exist non-negative integers μ_{ij} such that any two distinct blocks B and C with $B \in \mathscr{B}_i$ and $C \in \mathscr{B}_j$ intersect in exactly μ_{ij} points ($i, j \in \{1, \dots, c\}$).*

Proof. Denote by $\{\mathscr{B}_1, \dots, \mathscr{B}_c\}$ the strong resolution of $\mathbf{D}$. Let $p_1, \dots, p_v$ be the points of $\mathbf{D}$; the blocks of $\mathbf{D}$ are numbered in the following way:

$$\mathscr{B}_1 = \{B_1, \dots, B_{m_1}\}, \qquad \mathscr{B}_2 = \{B_{m_1+1}, \dots, B_{m_1+m_2}\}, \dots, \mathscr{B}_c = \{B_{b-m_c}, \dots, B_b\}.$$

Let A be the incidence matrix of $\mathbf{D}$ according to the above numeration. The

$(v+c)\times(b+1)$-matrix A_1 is defined as follows:

$$A_1=\left(\begin{array}{cccc|c}
 & & A & & 1\\
 & & & & 1\\
 & & & & \vdots\\
 & & & & 1\\
\hline
\underbrace{11\cdots1}_{m_1} & & & & 0\\
 & \underbrace{11\cdots1}_{m_2} & & 0 & 0\\
0 & & \ddots & & \vdots\\
 & & & \underbrace{11\cdots1}_{m_c} & 0
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\}v\\
\left.\begin{array}{l}\\ \\ \\ \\ \end{array}\right\}c
\end{array}$$

Since $v+c=b+1$, A_1 is a quadratic matrix. We shall see that A_1 is non-singular and we shall compute the inverse of A_1 explicitly. In order to do this, it is convenient to define the following $b\times v$-matrix

$$J^+=\left(\begin{array}{ccc}
k_1 & \cdots & k_1\\
\vdots & & \vdots\\
k_1 & \cdots & k_1\\
 & \vdots & \\
k_c & \cdots & k_c\\
\vdots & & \vdots\\
k_c & \cdots & k_c
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}m_1\\
\\
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}m_c
\end{array}.$$

Step 1. $A_1B_1=I$, where B_1 is the following matrix:

$$B_1=\left(\begin{array}{c|ccc}
 & m_1^{-1} & & \\
 & \vdots & & \\
 & m_1^{-1} & & \\
\dfrac{1}{r-\lambda}A'-\dfrac{1}{v(r-\lambda)}J^+ & & \ddots & \\
 & & & m_c^{-1}\\
 & & & \vdots\\
 & & & m_c^{-1}\\
\hline
\dfrac{1}{v}\ \cdots\ \dfrac{1}{v} & \dfrac{-k_1}{v} & \cdots & \dfrac{-k_c}{v}
\end{array}\right)
\begin{array}{l}
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}m_1\\
\\
\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}m_c\\
\\
\end{array}.$$

Namely: We compute the element d_{ij} in position (i, j) of A_1B_1.

(1) $i, j \in \{1, \ldots, v\}$, $i = j$. Then

$$d_{ij} = d_{ii} = \sum_{s=1}^{c} \rho_s\left(\frac{1}{r-\lambda} - \frac{k_s}{v(r-\lambda)}\right) + \frac{1}{v} = \frac{r}{r-\lambda} - \frac{\lambda(v-1)+r}{v(r-\lambda)} + \frac{1}{v} = 1.$$

(2) $i, j \in \{v+1, \ldots, v+c\}$, $i = j$. It follows that

$$d_{ij} = d_{ii} = \sum_{s=1}^{m_i} m_i^{-1} = 1.$$

(3) $i, j \in \{1, \ldots, v\}$, $i \neq j$. Denote by λ_{sij} the number of blocks in the class $\mathcal{B}_s$ joining the points p_i and p_j. Clearly, $\sum_{s=1}^{c} \lambda_{sij} = \lambda$. Therefore

$$d_{ij} = \sum_{s=1}^{c} \left[\lambda_{sij}\left(\frac{1}{r-\lambda} - \frac{k_s}{v(r-\lambda)}\right) + (\rho_s - \lambda_{sij})\frac{-k_s}{v(r-\lambda)}\right] + \frac{1}{v}$$

$$= \frac{\lambda}{r-\lambda} - \frac{1}{v(r-\lambda)}\sum_{s=1}^{c} \rho_s k_s + \frac{1}{v} = 0.$$

(Note that holds:

$$\sum_{s=1}^{c} \rho_s k_s = \sum_{s=1}^{c} \rho_s(k_s - 1) + \sum_{s=1}^{c} \rho_s = \lambda(v-1) + r.)$$

(4) $i, j \in \{v+1, \ldots, v+c\}$, $i \neq j$. In this case, obviously $d_{ij} = 0$.

(5) $i \in \{1, \ldots, v\}$, $j \in \{v+1, \ldots, v+c\}$. Using $\rho_j v = m_j k_j$ we get

$$d_{ij} = \rho_j \frac{1}{m_j} - \frac{k_j}{v} = 0.$$

(6) $j \in \{1, \ldots, v\}$, $i \in \{v+1, \ldots, v+c\}$. In this case we see

$$d_{ij} = \rho_i\left[\frac{1}{r-\lambda} - \frac{k_i}{v(r-\lambda)}\right] + (m_i - \rho_i)\frac{-k_i}{v(r-\lambda)} = 0.$$

Thus, Step 1 is proved. In particular, A_1 is regular and $B_1A_1 = I$.

Step 2. The number of points on two distinct blocks B and C depends only on the classes B and C belong to.

Namely: First, let B_i and B_j be two distinct blocks in the same class $\mathcal{B}_s$. If c_{ij} denotes the element of B_1A_1 in position (i, j), we get

$$0 = c_{ij} = [B_i, B_j]\left(\frac{1}{r-\lambda} - \frac{k_s}{v(r-\lambda)}\right) + (k_s - [B_i, B_j])\frac{-k_s}{v(r-\lambda)} + \frac{1}{m_s}.$$

Therefore, $[B_i, B_j]$ is independent from the choice of B_i, $B_j \in \mathcal{B}_s$.

Now denote by B_i and B_j two blocks in different classes, say $B_i \in \mathcal{B}_s$ and $B_j \in \mathcal{B}_t$. Then

$$0 = c_{ij} = [B_i, B_j]\left(\frac{1}{r-\lambda} - \frac{k_s}{v(r-\lambda)}\right) + (k_t - [B_i, B_j])\frac{-k_s}{v(r-\lambda)}.$$

So, $[B_i, B_j] = k_s k_t / v$ depends only on s and t.

Together, Proposition 2 is proved. □

Lemma 1. *Denote by* D *an* (r,λ)*-design. Suppose that there exists a partition* $\{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$ *of the block set of* D *satisfying property* $(*)$. *Then any two blocks of the same class are incident with the same number of points.*

Proof. We denote by m_i the number of blocks in $\mathcal{B}_i$ ($i \in \{1, \ldots, c\}$). Let B be a block in the class $\mathcal{B}_s$. Counting the incidences (x, X) with $x\,I\,B$ and $X \neq B$ we get

$$[B](r-1) = \sum_{\substack{i=1\\ i\neq s}}^{c} m_i \mu_{is} + (m_s - 1)\mu_{ss}. \quad \Box$$

Lemma 2. *Let* D *be an* (r,λ)*-design having a resolution* $\{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$ *with parameters* $\rho_1, \ldots, \rho_c$ *satisfying* $(*)$. *Denote by* k_i *the number of points on a block of* $\mathcal{B}_i$. *Then*

$$\mu_{ii} = k_i - r + \lambda \qquad (i \in \{1, \ldots, c\}).$$

Proof. Denote by B a block in the class $\mathcal{B}_i$. Let p be a point on B and let q be a point off B. Counting in two ways the incidences (x, X) with $x\,I\,B$, $x \neq p$, $p\,I\,X$ and $X \neq B$ (or, $x\,I\,B$, $q\,I\,X$, respectively) it follows that

$$(k_i - 1)(\lambda - 1) = (\rho_i - 1)(\mu_{ii} - 1) + \sum_{\substack{j=1\\ j\neq i}}^{c} \rho_j(\mu_{ij} - 1)$$

(or, $k_i\lambda = \sum_{j=1}^{c} \rho_j \mu_{ij}$, respectively).

Subtracting these two equations, the assertion follows. □

Proposition 3. *Let* D *be an* (r,λ)*-design. Suppose that* D *has a resolution* $\{\mathcal{B}_1, \ldots, \mathcal{B}_c\}$ *satisfying* $(*)$. *Then* $b + 1 = v + c$.

Proof. Let $\rho_1, \ldots, \rho_c$ be the parameters of the resolution in question. We already know the following equations:

$$\mu_{ii} = k_i - r + \lambda \qquad (i \in \{1, \ldots, c\}), \tag{1}$$

$$v\rho_i = m_i k_i \qquad (i \in \{1, \ldots, c\}), \tag{2}$$

$$\sum_{i=1}^{c} \rho_i k_i = \lambda(v-1) + r. \tag{3}$$

Denote by B a block of $\mathcal{B}_i$. Counting the incidences (x, X) with $x\,I\,B$, $x \in \mathcal{B}_i$ and $X \neq B$ in two ways we get

$$k_i(\rho_i - 1) = (m_i - 1)\mu_{ii}. \tag{4}$$

Hence, by (2) and (1) it follows

$$k_i\frac{m_ik_i-v}{v}=k_i(\rho_i-1)=(m_i-1)\mu_{ii}=(m_i-1)(k_i-r+\lambda),$$

or

$$m_ik_i^2-m_ik_iv=v(m_i-1)(\lambda-r).$$

Consequently,

$$\sum_{i=1}^{c} m_ik_i^2-v\sum_{i=1}^{c} m_ik_i=v(\lambda-r)\sum_{i=1}^{c}(m_i-1),$$

or, in view of (2),

$$v\sum_{i=1}^{c}\rho_ik_i-v^2\sum_{i=1}^{c}\rho_i=v(\lambda-r)\sum_{i=1}^{c}(m_i-1).$$

Using (3) we get

$$\lambda(v-1)+r-vr=(\lambda-r)(b-c),\quad \text{or}\quad (v-1)(\lambda-r)=(\lambda-r)(b-c),$$

hence $b+1=v+c$. □

Remark. By Propositions 1, 2, and 3, Theorem 1 is proved.

4. Examples

Let $\{\mathcal{B}_1,\ldots,\mathcal{B}_c\}$ be a resolution of the (r,λ)-design $\mathbf{D}=(\not p,\mathcal{B},I)$. The *complement of* $\mathbf{D}$ *with respect to* $\mathcal{B}_1$ is the incidence structure $\mathbf{D}^c(\mathcal{B}_1)=(\not p,\mathcal{B},I')$, where I' is defined as follows:

$$p\,I'\,B \quad\Leftrightarrow\quad p\,I\,B \quad \text{for } p\in\not p \text{ and } B\in\mathcal{B}-\mathcal{B}_1,$$

$$p\,I'\,B \quad\Leftrightarrow\quad p\,\not I\,B \quad \text{for } p\in\not p \text{ and } B\in\mathcal{B}_1.$$

Lemma 3. (a) $\mathbf{D}^c(\mathcal{B}_1)$ *is an* (r',λ')*-design and* $\{\mathcal{B}_1,\ldots,\mathcal{B}_c\}$ *is a resolution of* $\mathbf{D}^c(\mathcal{B}_1)$.

(b) *If* $\{\mathcal{B}_1,\ldots,\mathcal{B}_c\}$ *is a strong resolution of* $\mathbf{D}$, *then it is a strong resolution of* $\mathbf{D}^c(\mathcal{B}_1)$ *as well.*

Proof. Let $\rho_1,\ldots,\rho_c$ be the parameters of the resolution $\{\mathcal{B}_1,\ldots,\mathcal{B}_c\}$ and denote by m_i the number of blocks in $\mathcal{B}_i$ ($i\in\{1,\ldots,c\}$).

(a) Obviously, in $\mathbf{D}^c(\mathcal{B}_1)$ through any point there are exactly $\rho_1'=m_1-\rho_1$ blocks of $\mathcal{B}_1$. If $\lambda_{p,q}$ is the number of blocks in $\mathcal{B}_1$ which are (in $\mathbf{D}$) incident with the two distinct points p and q, then there are exactly $m_1-2\rho_1+\lambda_{p,q}$ blocks of $\mathcal{B}_1$ which are (in $\mathbf{D}^c(\mathcal{B}_1)$ incident with p and q. Hence $\mathbf{D}^c(\mathcal{B}_1)$ is an $(m_1-\rho_1+\rho_2+\cdots+\rho_c,\lambda+m_1-2\rho_1)$-design. Moreover, $\{\mathcal{B}_1,\ldots,\mathcal{B}_c\}$ is a resolution with parameters $m_1-\rho_1,\rho_2,\ldots,\rho_c$ of $\mathbf{D}^c(\mathcal{B}_1)$.

(b) In $D^c(\mathcal{B}_1)$, any two distinct blocks of $\mathcal{B}_1$ intersect in exactly $v-2k_1+\mu_{11}$ points, while any blocks of $\mathcal{B}_1$ has exactly $v-k_1-k_i+\mu_{1i}$ points in common with any block of $\mathcal{B}_i$ $(i\in\{2,\ldots,c\})$.

Thus, by Theorem 1, $D^c(\mathcal{B}_1)$ is strongly resolvable. □

Remark. (1) The above lemma says that we must only consider strongly resolvable (r, λ)-designs in which for any block B it holds $[B]\leqslant\frac{1}{2}v$.

(2) Lemma 3 gives us a construction method: Take a strongly resolvable (r, λ)-design D. (In particular, one can take a strongly resolvable block design!) Then the complement of D with respect to any number $s\leqslant c$ of resolution classes is a strongly resolvable (r', λ')-design.

Theorem 2. *Let D be a strongly ρ-resolvable (r, λ)-design. Then*

(a) *D has at most two distinct block lengths.*

(b) *If any block has at most $\frac{1}{2}v$ points, then D is a strongly resolvable block design.*

Proof. In our situation, the equations (1), (2) and (4) of the preceding section imply

$$k_i(\rho-1)=(m_i-1)\mu_{ii}=(m_i-1)(k_i-r+\lambda)$$
$$=m_ik_i-m_i(r-\lambda)-k_i+r-\lambda=v\rho-\frac{v\rho}{k_i}(r-\lambda)-k_i+r-\lambda,$$

or

$$k_i^2\rho=v\rho k_i-v\rho(r-\lambda)+(r-\lambda)k_i,$$

i.e.

$$k_i=\frac{1}{2\rho}[v\rho+r-\lambda\pm\sqrt{(v\rho+r-\lambda)^2-4v\rho^2(r-\lambda)}].$$

Thus, there are only two possible values of k_i. This proves (a).

(b) Since

$$\frac{v\rho+r-\lambda+\sqrt{\ }}{2\rho}>\frac{v}{2},$$

under our present assumptions, any block of D has the same size. □

Now, we shall give a general construction method.

Theorem 3. *Let $D=(\mathcal{P}, \mathcal{B}, I)$ be a symmetric 2-(v, k, λ) block design. Suppose that there is a non-empty subset $\mathcal{M}$ of $\mathcal{P}$ with the following properties:*

(a) *Any block of D contains all points of $\mathcal{M}$ or is incident with exactly m points of $\mathcal{M}$.*

(b) *For any two blocks B and C of D it holds: Either B and C intersect $\mathcal{M}$ in the same point set, or B and C have exactly n points of $\mathcal{M}$ in common.*

Then there exists a strongly resolvable (k, λ)-design with exactly $M+1$ classes, where M denotes the number of points in m.

Proof. Clearly, under conditions (a) and (b), $M>m>n$ (*otherwise* all blocks would pass through a common point).

We define the set $\mathscr{S}$ by

$$\mathscr{S}=\{(B)\cap m \mid B\in\mathscr{B}, m\nsubseteq (B)\}.$$

Let $\mathscr{S}$ have exactly $c-1$ elements: $\mathscr{S}=\{S_1,\ldots,S_{c-1}\}$. Now we define the incidence structure $\mathbf{D}-m$ as follows:

The *points* of $\mathbf{D}-m$ are the points in $p-m$; the *blocks* of $\mathbf{D}-m$ are the blocks of $\mathbf{D}$; the *incidence relation* of $\mathbf{D}-m$ is induced by $\mathbf{D}$.

Since any point of the symmetric 2-(v, k, λ) design $\mathbf{D}$ is incident with k blocks, $\mathbf{D}-m$ is a (k, λ)-design. We define a partition $\{\mathscr{B}_1,\ldots,\mathscr{B}_c\}$ of $\mathscr{B}$ in the following natural way:

$$\mathscr{B}_i=\{B \mid B\in\mathscr{B}, (B)\cap m=S_i\} \quad \text{for } i\in\{1,\ldots,c-1\},$$

and

$$\mathscr{B}_c=\{B \mid B\in\mathscr{B}, m\subseteq (B)\}.$$

We claim that $\{\mathscr{B}_1,\ldots,\mathscr{B}_c\}$ is a strong resolution of $\mathbf{D}-m$.

First of all, let us show that $\{\mathscr{B}_1,\ldots,\mathscr{B}_c\}$ is a resolution of $\mathbf{D}-m$. For a fixed point p of $\mathbf{D}-m$, denote by ρ' the number of blocks through p containing m. Counting the incidences (x, X) with $x\in m$ and $p\,I\,X$ we get

$$\lambda M=\rho' M+(k-\rho')m, \quad \text{or} \quad \rho'(M-m)=\lambda M-km.$$

Since $M>m$, ρ' is independent from the choice of p.

Now, for a fixed $i\in\{1,\ldots,c-1\}$, let ρ be the number of blocks through p in the class $\mathscr{B}_i$. We count the number of incidences (x, X) with $x\in S_i$, $p\,I\,X$ and $m\nsubseteq (X)$:

$$m(\lambda-\rho')=\rho m+(k-\rho'-\rho)n, \quad \text{i.e.} \quad \rho(m-n)=m(\lambda-\rho')-(k-\rho')n.$$

Since $m>n$, ρ is independent from the choice of p and $i\in\{1,\ldots,c-1\}$.

Clearly, $\rho>0$, since otherwise any block of $\mathbf{D}$ would pass through m. Next we claim $\rho'>0$. (*Otherwise*, $\mathbf{D}-m$ were a 2-$(v-M, k-m, \lambda)$ block design with $b'=v$ blocks and $r'=k$ blocks through any point. This would imply

$$k(k-m-1)=r'(k-m-1)=(v-M-1)\lambda$$
$$=(v-1)\lambda-M\lambda=k(k-1)-M\lambda,$$

i.e., $M\lambda=km$. Moreover,

$$v=b'=(v-M)k/(k-m),$$

or $Mk=mv$. Together we would have

$$\frac{k}{\lambda}=\frac{M}{m}=\frac{v}{k},$$

i.e. $v\lambda = k^2$. Consequently,

$$k(k-1) = (v-1)\lambda = k^2 - \lambda,$$

so $k = \lambda$: a contradiction, since in a symmetric block design any two distinct blocks intersect in $\lambda < k$ points.)

Now, let us consider the intersection numbers of $\mathbf{D} - \mathcal{m}$. Since $\mathbf{D}$ is a symmetric block design, any two distinct blocks intersect in exactly λ points of $\mathbf{D}$. Therefore,

$$\begin{aligned}
\mu_{ii} &= \lambda - m \quad \text{for } i \in \{1, \ldots, c-1\},\\
\mu_{cc} &= \lambda - M,\\
\mu_{ij} &= \lambda - n \quad \text{for } i, j \in \{1, \ldots, c-1\} \text{ and } i \neq j,\\
\mu_{ic} &= \lambda - m \quad \text{for } i \in \{1, \ldots, c-1\}.
\end{aligned}$$

Thus, the intersection numbers depend only on the classes, the blocks in question belong to. Therefore, by Theorem 1, $\mathbf{D} - \mathcal{m}$ is strongly resolvable.

Since $\mathbf{D} - \mathcal{m}$ has exactly $v - M$ points, v blocks and c classes, by Theorem 1 it follows that

$$v + 1 = v - M + c,$$

i.e., $c = M + 1$. $\square$

Remark. Consider the incidence structure $\mathbf{D}(\mathcal{m}) = (\mathcal{m}, \mathcal{S}, \in)$. If $\mathbf{D}(\mathcal{m})$ is a symmetric block design, then the conditions (a) and (b) are fulfilled. On the other hand, if $\mathbf{D} - \mathcal{m}$ is strongly resolvable, then $\mathbf{D}(\mathcal{m})$ is an incidence structure in which any two distinct blocks intersect in a constant number of points (namely n); moreover, the number of points (namely M) equals the number of blocks (namely $c-1$). Using Theorem 1.1 of Ryser [3], it follows that $\mathbf{D}(\mathcal{m})$ is a symmetric block design or a so-called 'n-design'.

Examples. (1) If there exists a symmetric 2-(v, k, λ) block design $\mathbf{D}$, then there exists a strongly resolvable (k, λ)-design with two resolution classes.

For: Take as $\mathcal{m}$ a 1-point set.

(2) For any prime-power q and any two integers t, d with $0 \leq t \leq d-2$ there exists a strongly resolvable $(q^{d-1} + \cdots + q + 1, q^{d-2} + \cdots + q + 1)$-design with exactly $q^t + \cdots + q + 2$ resolution classes.

Namely: Consider a projective space $\mathbf{P}$ of order q and dimension d. Let $\mathbf{U}$ be a t-dimensional subspace of $\mathbf{P}$. Then the system $\mathbf{D} = \mathbf{P}_{d-1}(d, q)$ of points and hyperplanes of $\mathbf{P}$ is a symmetric block design, and $\mathbf{U}$ is a subset of points fulfilling properties (a) and (b). Thus $\mathbf{D} - \mathbf{U}$ is a strongly resolvable (r, λ)-design with the parameters in question.

By the method indicated in [1], 2.4.36, one can easily construct many non-isomorphic strongly resolvable (r, λ)-designs with the above parameters.

(3) If there exist Hadamard matrices of order $a = 2s$ and $b = 2t$, then there is a strongly resolvable $(2st - 1, st - 1)$-design with exactly a resolution classes.

For: Let A and B be normalized Hadamard matrices of order a and b, respectively. Consider the direct product $H = A \times B$ of A and B. It is well known that H is also a normalized Hadamard matrix. Moreover, as A is normalized, the first b rows of H consist of a copies of B. Since B is normalized, any column of B contains $\frac{1}{2}b$ or b ones. Thus, any column of H has exactly $\frac{1}{2}b$ or b ones in the first b rows.

In other words: Let $\mathbf{D}$ be the block design corresponding to H, and denote by $\mathcal{M}$ the set of points of $\mathbf{D}$ corresponding to row $2, 3, \ldots, b$ of H. Then $\mathcal{M}$ satisfies the hypotheses of Theorem 3.

We conclude with the following question: In all examples of strongly resolvable (r, λ)-designs constructed by Theorem 3 and Lemma 3, there are at most four blocks lengths. Is there a number z such that any strongly resolvable (r, λ)-design has at most z block lengths?

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